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Axiomatic Logics for *ATIS*

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The Argument for a Symbolic Logic

Elizabeth Steiner, in her book *Methodology of Theory Building*[1], asserts:

> One must understand the many forms (kinds) of theory if one is not to apply the wrong art, i.e., if one is not to criticize or construct theory erroneously.

This same word of caution needs to be applied to the choice of logic that underlies the development of theory. The logic of a theory provides the means by which validity of statements of the theory can be "proved" as "true," and provides the means by which valid statements of the theory are derived.

For a scientific theory, normally a *symbolic logic*; that is, *formal logic,* is desired as such provides a means to obtain rigorous proofs for the validity of statements.

The logic required for *ATIS* will be an adaptation of the *Sentential Calculus* and *Predicate Calculus* that is normally used for mathematics and the mathematical sciences. While both calculi are concerned with analyzing statements based only on the form of the statements, they differ in terms of the types of statements analyzed. The *Sentential Calculus* is concerned with the form of the aggregate statement with no concern of what is contained within the statement. The *Predicate Calculus* is concerned with the logic of predicates; that is, statements and their constituent parts, as related to quantifiers—normally the universal and existential quantifiers, although others will be required for the logic of *ATIS*.

The advantage of a symbolic logic is that proofs are dependent only on the form of the statements, and not on their content. The advantage is that while it may take great insight to discover a theorem, once discovered it can be checked very systematically. The emphasis for theory development, however, is that the theoretician must continue to rely on intuition as the primary means of theory development, and the rigors of the basic logic are but a tool to assist in this development.

Steiner defines 'intuition' as a "non-inferential form of reasoning. It is a direct intellectual observation of the essence of what is given in experience."[2]

As will be discussed later, the System Logic schemas will be presented in two forms: Those that are derived directly from the axioms and should, therefore, be considered directly descriptive of the system, and those that are "theory construction axioms" and are, therefore, to be evaluated through intuition or other analytic tools before being considered part of the theory. The definition of 'intuition' by Charles Sanders Peirce addresses this desired theory-building method very directly when he states:

> Intuition is the regarding of the abstract in a concrete form, by the realistic hypostatization of relations.[3]

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[1] *Methodology of Theory Building*, Elizabeth Steiner, Indiana University, Educology Research Associates, Sydney, 1988.

[2] *Methodology*, *Ibid*., p. 93.

[3] *Collected Papers of Charles Sanders Peirce*, Volume 1, *Principles of Philosophy*, (Editors) Charles Hartshorne and Paul Weiss, The Belknap Press of Harvard University Press, Cambridge (1960), §1.383, p. 203.

While a schema can checked by following well-defined steps, a pragmatic logic must guide the development and acceptance of the theory. The need for a pragmatic logic is especially relevant for *ATIS System Construction Theorems* (*SCTs*) that are an integral part of the theory explication. The far-reaching consequences of the introduction of this theory-development methodology is not elsewhere discussed in the literature, as far as this researcher has been able to determine, and will be only referenced herein since there may be important proprietary consequences resulting from its usage. Essentially, the value of such theorem schemas will depend on the rules of construction that are defined for their usage. However, they will be further considered in a later section, to as great a degree as possible, in the section entitled *Significance of SCTs*.

While 'formal logic' is frequently assumed, we will state precisely what is meant by such logic. For our definition, 'symbol', 'language', 'formation', and 'transformation' will be taken as primitive terms. Then, *formal logic* is defined as follows, where "$=_{df}$" is read "*is defined as*":

Formal Logic $=_{df}$ A language that contains—

(1) Symbols,

(2) Well-formed formulas derived from the symbols as determined by formation rules,

(3) Axioms that are selected well-formed formulas, and

(4) Transformation rules, normally consisting of only one, Modus Ponens.[4]

The *Predicate Calculus* can be either a first-order or higher-order logic.

In *first-order logic*, quantification covers only individual elements (components) of a specific type or class; that is, only elements of a well-defined set (class) are considered. First-order logic results in verifying properties of a class or subclass of elements.

In *second-order logic*, quantification covers predicates. Second-order logic results in verifying properties of a class or subclass of predicates.

In higher-order logic, quantification covers predicate formulas. Higher-order logic results in verifying properties of a class or subclass of predicate formulas.

Whereas the *Sentential* and *Predicate Calculi* provide the logical foundation of the empirical sciences, such application must be done with care when extending that application to *ATIS*. In fact, however, the logic required for *ATIS* is less complex than that required for mathematics and the mathematical sciences, at least initially. The reason is that mathematics and the mathematical sciences must consider distinctions between "x's" that represent "unknown" and "variable" elements. The "unknown" uses are referred to as the "free" occurrences of x, and the "variable" uses are referred to as the "bound" occurrences of x. In *ATIS*, only "bound" occurrences of x will be required.

For this reason, many of the problems encountered by mathematicians relating to the *Predicate Calculus* will not be a problem in the logical analyses of *ATIS*. The reason is that, as noted above, *ATIS* does not consider any statement with free occurrences of x; that is, there are no "unknowns." As will be seen, statements with "unknowns" in *ATIS* are non-sense. For *ATIS* all uses of x are bound; that is, they are variables.

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[4] It is noted that some treatments of a formal logic will also include Generalization as a transformation rule; however, in our logic Generalization is obtained as a theorem.

In *ATIS*, problems are not being solved in which an unknown is being sought, but what is being sought are the system relations that are true for all described components of a system. The problem with seeking unknowns in the type of statements that are being considered is that it is difficult, if not impossible, to assign any proper meaning to such statements.

For example, the following is a bound occurrence of x:

For $x \in \mathfrak{S}$,

$\forall x(I_{\wp}^{\uparrow}(x) \supset {}_{\mathfrak{S}}\mathscr{F}^{\downarrow}(x))$; that is, "If input increases, then filtration decreases."

However, '$I_{\wp}^{\uparrow}(x)$' may or may not make sense when x is an element of just any unknown system, or even within a known system. That is, let '$I_{\wp}^{\uparrow}(x)$' be a translation of "x is the increasing input of the toput subsystem." While this English sentence is grammatically correct and has a recognizable meaning, its meaning within *ATIS* is highly suspect, since the x is now an unknown, or simply fanciful. Even if x can be construed as the input of a toput subsystem, x cannot be construed as "increasing" since it is but a single component. Or if it can be construed as increasing, then there are other assumptions of which we are not informed. x in this context is considered an unknown, or is a free occurrence of x. It is a situation in which we would have to determine under what conditions and in which systems this statement would have a proper meaning. Such statements are precluded from *ATIS* analyses.

Intentional and Complex Systems

SIGGS Theory has been developed with a strong reliance on formal theory. The formal theories of concern are symbolic logic and mathematics. This report will explicate the symbolic logic that is used to explicate *ATIS*.

In order to be selective of our logic, its application must be understood. The types of systems with which we are concerned are *Intentional Complex Systems*.

***Intentional Systems*:** Intentional Systems *are ones that are goal-oriented, or that have "intended" outcomes.*

For the analyst of general systems, an *Intentional System* is one that is predictable within certain parameters; that is, its *behavior* is predictable under certain system component relations. The challenge is to determine which system component relations are predictable and what outcomes are obtained as a result of those relations.

The problem of selecting a specific logic on which to base an analysis of general systems is that such systems are *Complex Systems*.

***Complex Systems*:** Complex Systems *are systems that are defined by large numbers of components with a large number of multiple types of heterarchy connections (affect relations) that determine the behavior of the system and such behavior is distinct from the behavior of the individual system components.*

The challenge here is to develop an analysis that can actually analyze a very large number of relations with multiple types of relations.

Complex Systems: Shown above are three examples of complex systems.
The complexity is not only in terms of the people shown, each one being a complex system, but also the environment in terms of the foliage, structures, pottery, etc.
(Photographs by Kenneth R. Thompson)

In general, it has been concluded that such systems cannot be analyzed with linear logics, such as logics founded on implication and Modus Ponens, as are the *Sentential* and *Predicate Calculi*. However, such conclusions have been founded on the beliefs that systems cannot be analyzed that have multiple relations. Such is not the case.

Yi Lin[5] has defined systems with multiple relations. It is just such systems that are required for an analysis of *ATIS*. Further, however, the assumption that the *ATIS Predicate Calculus* is linear is misplaced. By reference only, it is recognized that an APT analysis has been incorporated into the evaluation of this systems theory, an analysis that is non-linear. The significance of this analysis resulting in a *Sentential Calculus* that is non-linear will be considered at a later time. Further, of significance to an APT analysis is that an *Axiomatic Temporal Implication Logic* has been developed that may be of value to APT and its integration with *ATIS* to develop a non-linear logic.

What is required for now is a formal method to analyze general systems, a symbolic logic and mathematical logic that formally express the properties and relations of a system such as system behavior, system structure, dynamic states, morphisms, etc.

The *Sentential Calculus* is frequently defined in terms of truth tables that provide a truth-functional analysis of statements. However, since *ATIS* is defined as an *Axiomatic Theory of Intentional Systems*, we will approach both the *Sentential Calculus* and the *Predicate Calculus* as axiomatic theories. Such an approach lends itself to clear statements of theorems and proofs. Further, such axiomatic logics are required since truth-table logics cannot address statements in general, and the complex statements of *ATIS*.

Before presenting the axioms of the theory, a brief overview will help to transition from the truth table approach to the axiomatic approach. For example, consider the Axiomatic Temporal Implication Logic shown on the next page.

5 Lin, Yi (1999). *General Systems Theory: A Mathematical Approach*. Kluwer Academic/Plenum Publishers, NY.

Axiomatic Temporal Implication Logic

Axiomatic Temporal Implication Logic

Temporal Implication Logic has been developed to address the logic with respect to empirical systems that have a time set and, therefore, a sequence of events. The types of relations that are of concern in this logic are those where one event precedes another in time, and the first is considered to imply the other. For example, the situation where feedin precedes feedout and there is a relation between the two that we wish to represent by an implication would fall within this classification.

Using conventional logic, paradoxes will arise whereby equivalences will result in the conclusion implying the premise, an empirical impossibility since the conclusion is subsequent in time to the premise. *Temporal Implication Logic* is designed to constructively handle temporal parameters of implication. To distinguish *Temporal Implication Logic* from the implication of the Sentential and Predicate Calculi, a distinctive symbol will be used. Whereas 'implication' for the Sentential and Predicate Calculi is normally designated as '$\supset$', *Temporal Implication*, TI, will be designated by '$\sqsupset$'.

The problem with TI is that equivalences are not valid when either predicate of the TI is negated. All other logical operations and equivalences hold. The following *Axiomatic Temporal Implication Logic* provides the logic required to formally prove theorems in an empirical theory where temporal implications occur. For this logic, the operation for negation is not allowed, while all other operations can be defined in terms of '$\sqsupset$' and '$\wedge$', which are the two basic undefined operations.

For the following axioms, $\mathcal{F}$, $\mathcal{P}$, $\mathcal{Q}$, and $\mathcal{R}$ are statements, and x is a variable; i.e., a bound occurrence of x.

TI-A.1.	$\mathcal{P} \sqsupset \mathcal{P}\mathcal{P}$
TI-A.2.	$\mathcal{P}\mathcal{Q} \sqsupset \mathcal{P}$
TI-A.3.	$(\mathcal{P} \sqsupset \mathcal{R}) \sqsupset (\mathcal{P} \sqsupset (\mathcal{Q} \sqsupset \mathcal{R}))$
T-A.4.	$\forall x(\mathcal{P} \sqsupset \mathcal{Q}) \sqsupset (\forall x\mathcal{P} \sqsupset (\forall x\mathcal{Q})$
TI-A.5.	$\mathcal{P} \sqsupset \forall x\mathcal{P}$, if there are no free occurrences of x in $\mathcal{P}$; i.e., no unknowns.
TI-A.6.	$\forall x\mathcal{F}(x,y) \sqsupset \mathcal{F}(y,y)$

For the following axioms, $\mathcal{F}$, $\mathcal{P}$, $\mathcal{Q}$, and $\mathcal{R}$ are statements, and x is a variable; i.e., a bound occurrence of x. The distinction between this axiom set and that of the logic of the Sentential and Predicate Calculi is that the following axiom has been removed:

$$(\mathcal{P} \sqsupset \mathcal{Q}) \sqsupset (\sim(\mathcal{Q}\mathcal{R}) \sqsupset \sim(\mathcal{R}\mathcal{P}))$$

In place of the above axiom, the following has been used:

$$\text{TI-A.3.} \quad (\mathcal{P} \sqsupset \mathcal{R}) \sqsupset (\mathcal{P} \sqsupset (\mathcal{Q} \sqsupset \mathcal{R}))$$

This replacement effectively precludes $\sim\mathcal{Q} \sqsupset \sim\mathcal{P}$ as a logical equivalence of $\mathcal{P} \sqsupset \mathcal{Q}$. It also precludes numerous other equivalences in which negation of statements occur.

Following are the definitions of '$\vee$' and '$\equiv$'. The exclusive "or," '$\veebar$', cannot be defined within this *Temporal Implication Logic*.

$$\mathcal{P} \vee \mathcal{Q} =_{df} (\mathcal{P} \sqsupset \mathcal{Q}) \sqsupset \mathcal{Q} \qquad\qquad \mathcal{P} \equiv \mathcal{Q} =_{df} (\mathcal{P} \sqsupset \mathcal{Q}) \wedge (\mathcal{Q} \sqsupset \mathcal{P})$$

This logic is designed specifically to address the problems relating to temporal implications as distinct from the standard logic that does not address this issue.

In practice, when analyzing a specific system, both the TI Logic and Standard Logic will be utilized. For any time-dependent implication, the TI Logic will be used. For all other considerations, the Standard Logic will be used.

Symbolic Logic

Symbolic logic is a tool designed for scientific reasoning. In particular, it is a tool designed for *ATIS* reasoning, and also for educology reasoning; such reasoning required for a proper analysis of an *Education Systems Theory* (*EST*). It is by an interpretation of *ATIS* that educology is explicated, and by which an *EST* is retroduced.

It is through the use of symbolic logic that these theories are made precise and explicated. The main reason for using symbolic logic is that it is a means of obtaining precise definitions for the logical consequence of one statement from another. The main advantage of a formal logic is in being able to prove statements about a theory, and only minimally in being able to determine conclusions about the theory. Intuitive arguments are the more reliable source for obtaining answers, while formal arguments are required for proving those answers. For an empirical theory, like *EST*, answers are obtained by direct observation that have been predicted by the formal development of the theory and the intuitive arguments that such predictions are valid. That is, especially for the Theory-Construction Schemas, intuitive arguments are essential for guiding the application of the formal arguments.

One of the greatest advantages of a formal logic is that it provides a precise definition for determining when one statement is a logical consequence of another. When one comprehends the power of this advantage, then the fruitfulness of the predictive logic will be realized.

The objective of constructing a formal logic is that it will provide the precise criterion by which instances of *ATIS* reasoning will be determined as being correct. With this correct reasoning, one can confidently provide the predictions determined by the theory.

The logical consequence of one statement from another is obtained by a sequence of well-defined statements such that each statement is known to be valid; that is, is an axiom, is an assumption or is derived from previous statements of the sequence according to specific rules of inference.

Valid statements are only those that are axioms or are derived only from axioms.

Rules of inference are restricted to Modus Ponens.

The *ATIS* Sentential Calculus

The *ATIS Sentential Calculus* is a theory of statement formulas in which the statements are translations of sentences within *ATIS*. For *ATIS*, a statement is a declarative sentence that relates exclusively to system components, relations or properties of *ATIS*. While the *Sentential Calculus* herein considered may be equivalent to that used for mathematics and the mathematical sciences, it is important to note that the extended logic herein considered is that developed specifically for *ATIS*, and is not intended to be a logic generally utilized by mathematicians, although it may be applicable.

Statements will be expressed by capital letters; e.g., "$\mathcal{P}$," "$\mathcal{Q}$," etc., and are translations of their English sentences "A" and "B," respectively. All statement functions of the theory are derived from only two undefined functions: '∧' and '~', which are read "and" and "not," respectively. [**<u>NOTE</u>:** Other symbols than the ones shown may be used. For example, at times the symbol '·' is used in place of '∧'.]

Therefore, '$\mathcal{P} \wedge \mathcal{Q}$' is read "$\mathcal{P}$ and $\mathcal{Q}$," and is a translation of the English sentence "A and B"; and '$\sim\mathcal{P}$' is read "not $\mathcal{P}$," and is a translation of the negation of the English sentence "A". While we will read '$\sim\mathcal{P}$' as "not $\mathcal{P}$," the English sentence may take several forms depending on what is required to assert the negation of "A."

A statement formula is a string of statements combined with $\wedge$ and ~.

'$\wedge$' and '$\sim$' are the first two functions of the *Sentential Calculus*:

(1) $\mathcal{P} \wedge \mathcal{Q}$

(2) $\sim \mathcal{P}$

While these functions are undefined, they will be interpreted as having "truth values" ("validity values") defined by the following "truth-value tables" ("validity-value tables"). While the values are commonly thought of as "True" or "False," in fact they are but assigned values with no relation to "truth." Instead, to further emphasize their application to empirical theories, they will be interpreted as "valid" and "not-valid". This will be emphasized in the following tables by using '$\top$' for "valid" and '$\perp$' for "not-valid." The "validity table" then simply presents the four possible combinations of '$\top$' and '$\perp$' in the first table and the two possible combinations in the second.

Table 1: Validity table for the operation '$\wedge$'

$\mathcal{P}$	$\mathcal{Q}$	$\mathcal{P}$	$\wedge$	$\mathcal{Q}$
$\top$	$\top$	$\top$	$\top$	$\top$
$\top$	$\perp$	$\top$	$\perp$	$\perp$
$\perp$	$\top$	$\perp$	$\perp$	$\top$
$\perp$	$\perp$	$\perp$	$\perp$	$\perp$

Table 2: Validity table for the operation '$\sim$'

$\mathcal{P}$	$\sim\mathcal{P}$
$\top$	$\perp$
$\perp$	$\top$

As demonstrated in Tables 1 and 2, the operation '$\wedge$' takes the value '$\top$' only when both $\mathcal{P}$ and $\mathcal{Q}$ are $\top$; and the operation '$\sim$' takes the value that is the alternative to P.

By convention, '$\mathcal{P} \wedge \mathcal{Q}$' may be, and normally is, written as '$\mathcal{P}\mathcal{Q}$'.

'$\mathcal{P} \vee \mathcal{Q}$' ("$\mathcal{P}$ or $\mathcal{Q}$"—inclusive "or"; i.e., and/or), '$\mathcal{P} \veebar \mathcal{Q}$' ("$\mathcal{P}$ or $\mathcal{Q}$"—exclusive "or"; i.e., not both), '$\mathcal{P} \supset \mathcal{Q}$' ("$\mathcal{P}$ implies $\mathcal{Q}$" or "If $\mathcal{P}$ then $\mathcal{Q}$"), and '$\mathcal{P} \equiv \mathcal{Q}$' ("$\mathcal{P}$ if and only if $\mathcal{Q}$" or "$\mathcal{P}$ is equivalent to $\mathcal{Q}$") are defined as follows:

(3) $\mathcal{P} \vee \mathcal{Q} =_{df} \sim(\sim\mathcal{P} \sim\mathcal{Q})$

(4) $\mathcal{P} \veebar \mathcal{Q} =_{df} \sim(\sim\mathcal{P} \sim\mathcal{Q}) \wedge \sim(\mathcal{P}\mathcal{Q})$

(5) $\mathcal{P} \supset \mathcal{Q} =_{df} \sim(\mathcal{P} \sim \mathcal{Q})$

(6) $\mathcal{P} \equiv \mathcal{Q} =_{df} \sim(\mathcal{P} \sim \mathcal{Q}) \wedge \sim(\sim \mathcal{P}\mathcal{Q}) =_{df} (\mathcal{P} \supset \mathcal{Q})(\mathcal{Q} \supset \mathcal{P})$

These six functions are the ones by which the *Sentential Calculus* is explicated.

Since implication, ⊃, will be a very important function of the *ATIS Sentential Calculus*, its interpretation will be further considered. The function '$\mathcal{P} \supset \mathcal{Q}$' may be read in any one of the following ways, all of which are equivalent:

$\mathcal{Q}$ is a necessary condition for $\mathcal{P}$,

$\mathcal{P}$ is a sufficient condition for $\mathcal{Q}$,

$\mathcal{Q}$ if $\mathcal{P}$,

$\mathcal{P}$ only if $\mathcal{Q}$,

$\mathcal{P}$ implies $\mathcal{Q}$, and

If $\mathcal{P}$ then $\mathcal{Q}$.

Consider a list of statements, $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n$. Combine these statements by the use of '∧' and '~' in any manner desired, and call the result 'Γ'. As a result of this construction of Γ, Γ will be called a *statement formula*. A statement formula that is written using only '∧' and '~' will be defined as being in "standard form." The purpose of the *Sentential Calculus* is to determine when a statement formula is valid, and validity will be determined when the statement formula is "valid" pursuant to the validity-tables, or as a result of an axiomatic analysis for the *Sentential Calculus*.

Validity tables can assist in determining when a statement formula is valid regardless of the meaning of the statements that make up the formula. That is, in general, if the validity of the statements of a formula is unknown, then the validity of the formula cannot be determined.

However, statement formula validity can be determined, regardless of the validity of the statements, when the statement formula has a certain structure. For example, the statement formula "$\sim\mathcal{P}\mathcal{P}$" is always false and $\sim(\sim\mathcal{P}\mathcal{P})$ is always true regardless of the validity of $\mathcal{P}$. Validity tables can assist in determining under what conditions a statement formula is valid in the *Sentential Calculus*. Only one example will be provided since discussions on truth table analyses can be found in many introductory texts on logic.

As this is not intended to be a formal development of the *Sentential Calculus*, only the basic functions are shown as they are derived from '∧' and '~', and their validity tables will not be shown. The reader of this report is encouraged to take a course in formal logic that at least includes Venn Diagrams, Syllogisms, and "Truth Tables" (as commonly referred to in such courses). While these studies are a beginning for the comprehension of this report, it must be understood that the logic presented herein goes far beyond the scope of an introductory logic course.

Consider the following statement formula: $\Gamma = (\mathcal{P} \wedge \sim\mathcal{Q}) \vee [\mathcal{Q} \supset (\mathcal{P} \equiv \mathcal{Q} \wedge \sim\mathcal{P})]$.

Although operations other than '∧' and '~' are used in this statement formula, by the preceding definitions, they could be replaced with '∧' or '~' thus writing the statement formula in standard form as is required. This also demonstrates the need for such symbolic uses, as this statement formula in standard form would be:

~[~(𝒫 ~𝒬)~[~(𝒬 ~([~(𝒫 ~(𝒬 ~𝒫))~(~𝒫 (𝒬 ~𝒫))]))]].

In this statement formula, '𝒫' and '𝒬' are "parameters" of the formula. The question to be answered in the *Sentential Calculus* is under what conditions is this formula valid? We proceed as shown in the following tables. First, the possible values of 𝒫 and 𝒬 are entered, and the table is set up so as to have a column assigned for every statement and operation. While '~' could be given a separate column, it is normally less confusing to assign it to its associated statement. In these tables, '⊤' designates "valid," and '⊥' designates "not-valid." Starting with *Table 4*, the values shown in bold print are determined from the values in italics. The column designated as "5" in Table 3 under '∨' determines the values for the statement formula under the possible combinations of statement validity. By the grouping symbols, the values will be determined in the order designated in the last row.

Table 3: Assign values to "P" and "Q"

𝒫	𝒬	(𝒫	∧	~	∨	[𝒬	⊃	(𝒫	≡	𝒬	∧	~𝒫)]
⊤	⊤											
⊤	⊥											
⊥	⊤											
⊥	⊥											
		1	**2**	**1**	**5**	**1**	**4**	**1**	**3**	**1**	**2**	**1**

Table 4: Determine values of statements within the formula

𝒫	𝒬	(𝒫	∧	~	∨	[𝒬	⊃	(𝒫	≡	𝒬	∧	~𝒫)]
⊤	*⊤*	**⊤**		**⊥**		**⊤**		**⊤**		**⊤**		**⊥**
⊤	*⊥*	**⊤**		**⊤**		**⊥**		**⊤**		**⊥**		**⊥**
⊥	*⊤*	**⊥**		**⊥**		**⊤**		**⊥**		**⊤**		**⊤**
⊥	*⊥*	**⊥**		**⊤**		**⊤**		**⊥**		**⊥**		**⊤**

Table 5: Determine the values of the innermost operations

𝒫	𝒬	(𝒫	∧	~	∨	[𝒬	⊃	(𝒫	≡	𝒬	∧	~𝒫)]
⊤	⊤	⊤	⊥	⊥		⊤		⊤		⊤	⊥	⊥
⊤	⊥	⊤	⊤	⊤		⊥		⊤		⊥	⊥	⊥
⊥	⊤	⊥	⊥	⊥		⊤		⊥		⊤	⊤	⊤
⊥	⊥	⊥	⊥	⊤		⊥		⊥		⊥	⊥	⊤

Table 6: Determine the values of the second-level operation

𝒫	𝒬	(𝒫	∧	~	∨	[𝒬	⊃	(𝒫	≡	𝒬	∧	~𝒫)]
⊤	⊤	⊤	⊥	⊥		⊤		⊤	⊥	⊤	⊥	⊥
⊤	⊥	⊤	⊤	⊤		⊥		⊤	⊥	⊥	⊥	⊥
⊥	⊤	⊥	⊥	⊥		⊤		⊥	⊥	⊤	⊤	⊤
⊥	⊥	⊥	⊥	⊤		⊥		⊥	⊥	⊥	⊥	⊤

Table 7: Determine the values of the third-level operation

𝒫	𝒬	(𝒫	∧	~	∨	[𝒬	⊃	(𝒫	≡	𝒬	∧	~𝒫)]
⊤	⊤	⊤	⊥	⊥		⊤	⊥	⊤	⊥	⊤	⊥	⊥
⊤	⊥	⊤	⊤	⊤		⊥	⊤	⊤	⊥	⊥	⊥	⊥
⊥	⊤	⊥	⊥	⊥		⊤	⊥	⊥	⊥	⊤	⊤	⊤
⊥	⊥	⊥	⊥	⊤		⊥	⊤	⊥	⊥	⊥	⊥	⊤

Table 8: Determine the values for the statement formula

𝒫	𝒬	(𝒫	∧	~	∨	[𝒬	⊃	(𝒫	≡	𝒬	∧	~𝒫)]
⊤	⊤	⊤	⊥	⊥	⊥	⊤	⊥	⊤	⊥	⊤	⊥	⊥
⊤	⊥	⊤	⊤	⊤	⊤	⊥	⊤	⊤	⊥	⊥	⊥	⊥
⊥	⊤	⊥	⊥	⊥	⊥	⊤	⊥	⊥	⊥	⊤	⊤	⊤
⊥	⊥	⊥	⊥	⊤	⊤	⊥	⊤	⊥	⊥	⊥	⊥	⊤

Therefore, this statement formula is valid when 𝒫 is true and 𝒬 is false, or when both are false.

While the above example of a statement formula is rather simple, the determination of its validity is somewhat tedious, and was elaborated in six steps to make that very point. While "truth tables" can be used to determine the validity of many statement formulae, they are quite cumbersome when either long statement formulae are evaluated, or more than two different parameters are part of the formula. For example, eight rows of values are required for statements containing three distinct parameters, and 16 rows are required for formulas with four distinct parameters. In general, 2^n rows are required for n distinct parameters. An even more complex evaluation arises if; for example, a "contingent" value, C, is introduced. In this case, two parameters would require nine rows, and in general 3^n rows are required for n distinct parameters.

To see the efficacy of moving from validity table analyses to axiomatic analyses, we will evaluate the statement formula: $\mathcal{P} \supset [(\mathcal{P} \supset \mathcal{Q}) \supset \mathcal{Q}]$ as shown in Table 9.

Table 9: Statement formula, P ⊃ [(P ⊃ Q) ⊃ Q], analysis

$\mathcal{P}$	$\mathcal{Q}$	$\sim\mathcal{Q}$	$\supset$	$[(\mathcal{P}$	$\supset$	$\mathcal{Q})]$	$\supset$	$\sim\mathcal{P}]$
⊤	⊤	⊤	⊤	⊤	⊤	⊤	⊤	⊤
⊤	⊥	⊤	⊤	⊤	⊥	⊥	⊤	⊥
⊥	⊤	⊥	⊤	⊥	⊤	⊤	⊤	⊤
⊥	⊥	⊥	⊤	⊥	⊤	⊥	⊥	⊥

Modus Ponens

From Table 9, it is seen that the statement formula is a tautology; i.e., it takes the value "⊤" under any value of $\mathcal{P}$ and $\mathcal{Q}$. This statement formula is so important that it has been given the name "*Modus Ponens*." Expressed in analytic form, the statement formula is:

$$\mathcal{P}, \mathcal{P} \supset \mathcal{Q} \vdash \mathcal{Q}$$

This formula is read: "$\mathcal{P}$ and $\mathcal{P} \supset \mathcal{Q}$ yields $\mathcal{Q}$." This formula means that if you are given $\mathcal{P}$ and $\mathcal{P} \supset \mathcal{Q}$, you can conclude $\mathcal{Q}$.

Modus Ponens is applied when you have either proven that $\mathcal{P}$ and $\mathcal{P} \supset \mathcal{Q}$ are valid or are assumptions. The importance of Modus Ponens for our theory is that it provides the one and only logical rule for proving theorems.

Modus Talens

Another important statement formula is entitled "*Modus Talens*," and has the form $\sim\mathcal{Q} \supset [(\mathcal{P} \supset \mathcal{Q}) \supset \sim\mathcal{P}]$. The validity table for this statement formula is shown in Table 10.

Table 10: Statement formula, ~Q ⊃ [(P ⊃ Q) ⊃ ~P], analysis

$\mathcal{P}$	$\mathcal{Q}$	$\sim\mathcal{Q}$	$\supset$	$[(\mathcal{P}$	$\supset$	$\mathcal{Q})]$	$\supset$	$\sim\mathcal{P}]$
$\top$	$\top$	$\bot$	$\top$	$\top$	$\top$	$\top$	$\bot$	$\bot$
$\top$	$\bot$	$\top$	$\top$	$\top$	$\bot$	$\bot$	$\top$	$\bot$
$\bot$	$\top$	$\bot$	$\top$	$\bot$	$\top$	$\top$	$\top$	$\top$
$\bot$	$\bot$	$\top$	$\top$	$\bot$	$\top$	$\bot$	$\top$	$\top$

Modus Talens is used when you have $\sim\mathcal{Q}$ and $\mathcal{P} \supset \mathcal{Q}$. Given these two statement formulas, you can conclude ~P. Formally, this statement formula is: $\sim\mathcal{Q}, \mathcal{P} \supset \mathcal{Q} \vdash \sim\mathcal{P}$.

Numerous logical tautologies can be confirmed by the use of validity tables. Having done so, the results can be used without recourse to the validity tables. For example, by use of validity tables, it can be shown that '⊃' is a transitive operation. That is, if $\mathcal{P} \supset \mathcal{Q}$ and $\mathcal{Q} \supset \mathcal{R}$, then we can conclude that $\mathcal{P} \supset \mathcal{R}$. With the great number of axioms contained in *ATIS*, the transitivity operation greatly facilitates the proving of numerous theorems. Formally, this transitivity property is: $\mathcal{P} \supset \mathcal{Q}, \mathcal{Q} \supset \mathcal{R} \vdash \mathcal{P} \supset \mathcal{R}$.

Applying the transitivity property of '⊃' to the following SIGGS axioms the efficacy of such formal treatments of theories is seen once again.

Consider Axioms 144 and 150, stated as follows:

Axiom 144: If filtration decreases, then isomorphism increases.

Axiom 150: If automorphism increases, then input increases and storeput increases and fromput decreases and feedout decreases and filtration decreases and spillage decreases and efficiency decreases.

Stated formally, these axioms are:

Axiom 144: $\mathcal{FR}^{\downarrow} \supset \mathcal{I}^{\uparrow}$

Axiom 150: $\mathcal{A}^{\uparrow} \supset I_{\mathcal{P}}^{\uparrow} \wedge S_{\mathcal{P}}^{\uparrow} \wedge F_{\mathcal{P}}^{\downarrow} \wedge fo^{\downarrow} \wedge \mathcal{FR}^{\downarrow} \wedge \mathcal{SR}^{\downarrow} \wedge \mathcal{S}_{E}^{\downarrow}$

For Axiom 150, we will select only one of the conclusions, $\mathscr{R}^{\downarrow}$, to prove.

<u>*Given*</u>: Axiom 150: $\mathscr{A}^{\uparrow} \supset \mathscr{R}^{\downarrow}$; and

Axiom 144: $\mathscr{R}^{\downarrow} \supset \mathscr{I}^{\uparrow}$.

$\therefore\ \mathscr{A}^{\uparrow} \supset \mathscr{I}^{\uparrow}$

That is, from the transitivity of '⊃', we can obtain the theorem: $\vdash \mathscr{A}^{\uparrow} \supset \mathscr{I}^{\uparrow}$ from Axioms 150 and 144.

That is, if system automorphism increases, then isomorphism increases, and no validity tables are required to prove this theorem.

The purpose of a symbolic logic is to be able to consider the parameters of a theory without recourse to the meaning of the concepts, and the purpose of the *Sentential* and *Predicate Calculi* is to derive theorems based solely on the form of the statement formulas. This task of deriving theorems can be more easily accomplished by the use of an axiomatic approach to the *Sentential* and *Predicate Calculi*.

A note is required concerning the meaning of the symbols used above. Since they are part of a statement formula, they must be "statements." For example, the above formalization of Axiom 144 is of the statement: "If filtration decreases, then isomorphism increases." Formally, this is of the form "$\mathcal{P} \supset \mathcal{Q}$," where '$\mathcal{P}$' is "filtration decreases" and '$\mathcal{Q}$' is "isomorphism increases." However, there is more contained in these statements to actually make them "statements." They will be considered to read as follows: '$\mathcal{P}$' is "The system filtration decreases"; and '$\mathcal{Q}$' is "The system isomorphism increases," both of which are now declarative sentences about an *ATIS* system. Whenever a property is cited in an axiom, it is to be understood that the property is actually contained within a statement.

Axiomatic Sentential Calculus

Whereas validity tables are convenient for determining the validity of statement formulas, such tables cannot be generalized to all statements. To date, only an axiomatic method is known that is able to obtain validations of general statements. As a transition to the axioms required for validation of general statements, we will first consider a subset of those axioms, the validity-value axioms. These axioms will provide an excellent transition to axiomatic logic, since these axioms will produce those statements considered earlier, the statement formulae that can be validated by use of a validity table, and, therefore can be easily validated by two methods—validity tables and axioms.

In general, the axiomatic definition of valid statements is obtained by the following process: (1) Certain selected statements are called 'axioms' (and their selection may be somewhat arbitrary and may be modified to achieve certain objectives); (2) A transformation rule is selected, normally Modus Ponens (although other transformation rules are possible; for example, Generalization or Modus Talens); and (3) 'Valid statements' are those statements that are either axioms or can be derived from two or more axioms by successive applications of Modus Ponens.

It is worth mentioning again that only the form of the statements and not their meaning determines valid statements.

There are three axioms of the *Valid-Value Sentential Calculus* and one logical rule.

Let '$\mathcal{P}$', '$\mathcal{Q}$', and '$\mathcal{R}$' be statements of the theory, then—

The logical rule is Modus Ponens and the axiom schemas are:

(1) $\mathcal{P} \supset \mathcal{P}\mathcal{P}$

(2) $\mathcal{P}\mathcal{Q} \supset \mathcal{P}$

(3) $\mathcal{P} \supset \mathcal{Q} .\supset. \sim(\mathcal{Q}\mathcal{R}) \supset \sim(\mathcal{R}\mathcal{P})$

There are an infinite number of statements that will comprise the valid-value axioms; however, all axioms will be of one of the above three general forms, the *axiom schemas*. Further, all theorems of the *Valid-Value Sentential Calculus* can be derived from these three axioms and Modus Ponens.

A theorem will take the form: $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n \vdash \mathcal{Q}$, where the $\mathcal{P}$'s are statements and $\mathcal{Q}$ is an axiom, or $\mathcal{Q}$ is one of the $\mathcal{P}$'s, or $\mathcal{Q}$ is derived from the $\mathcal{P}$'s by repeated applications of Modus Ponens.

'$\vdash$' is read "yield", or in the case when we have only '$\vdash \mathcal{Q}$' it is read "yields $\mathcal{Q}$."

The theorem $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n \vdash \mathcal{Q}$ indicates that there is a sequence of statements, $\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_m$, called the *proof* of the theorem, such that S_m is $\mathcal{Q}$ and for each S_i, either:

(1) $\mathcal{S}_i$ is an axiom,

(2) $\mathcal{S}_i$ is a $\mathcal{P}$; i.e., an assumption,

(3) $\mathcal{S}_i$ is the same as some earlier $\mathcal{S}_j$, or

(4) $\mathcal{S}_i$ is derived from two earlier $\mathcal{S}$'s by Modus Ponens.

The sequence $\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_m$ is a proof within the Symbolic Logic so that $\mathcal{Q}$ is logically derived from the assumptions $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n$.

The following theorems are readily provable concerning $\vdash$:

Theorem. If $\mathcal{P}_1, \ldots, \mathcal{P}_n \vdash \mathcal{Q}$, then $\mathcal{P}_1, \ldots, \mathcal{P}_n, \mathcal{R}_1, \ldots, \mathcal{R}_m \vdash \mathcal{Q}$.

> Proof: Let $S_1, \ldots, S_s$ be the proof of $\mathcal{P}_1, \ldots, \mathcal{P}_n \vdash \mathcal{Q}$ where S_s is $\mathcal{Q}$. Clearly that same sequence will yield Q regardless of any additional assumptions.

Theorem. If $\mathcal{P}_1, \ldots, \mathcal{P}_n \vdash \mathcal{Q}_1$ and $\mathcal{Q}_1, \ldots, \mathcal{Q}_m \vdash \mathcal{R}$, then $\mathcal{P}_1, \ldots, \mathcal{P}_n, \mathcal{Q}_2, \ldots, \mathcal{Q}_m \vdash \mathcal{R}$.

Theorem. If $\mathcal{P}_1, \ldots, \mathcal{P}_n \vdash \mathcal{Q}_1$, $\mathcal{R}_1, \ldots, \mathcal{R}_m \vdash \mathcal{Q}_2$, and $\mathcal{Q}_1, \ldots, \mathcal{Q}_q \vdash \mathcal{S}$, then

$$\mathcal{P}_1, \ldots, \mathcal{P}_n, \mathcal{R}_1, \ldots, \mathcal{R}_m, \mathcal{Q}_3, \ldots, \mathcal{Q}_q \vdash \mathcal{S}.$$

Theorem. If $\vdash \mathcal{Q}_1$ and $\mathcal{Q}_1, \ldots, \mathcal{Q}_m \vdash \mathcal{R}$, then $\mathcal{Q}_2, \ldots, \mathcal{Q}_m \vdash \mathcal{R}$.

Theorem. If $\vdash \mathcal{Q}_1, \vdash \mathcal{Q}_2, \ldots, \vdash \mathcal{Q}_m$ and $\mathcal{Q}_1, \ldots, \mathcal{Q}_m \vdash \mathcal{R}$, then $\vdash \mathcal{R}$.

Since our main concern is to provide the means to explicate *ATIS*, the *Sentential Calculus* will not be further explicated. The following *List of Logical Schemas* is provided to facilitate the explication of the theories. This list is not exhaustive, but does represent those schemas that lend themselves to a fruitful explication of the theories. Following this list, the schemas will be used to demonstrate the value of such a symbolic logic by providing proofs of theorems. It is noted that technically these schemas are not actually part of the *Sentential Calculus* but are part of the metatheory, the *Meta-Sentential Calculus*. They are statements about the calculus that define the form or schemas that the theorems of the theory actually take.

List of Logical Schemas

The following list of the logical schemas is provided to facilitate the proof of theorems. The proof of various theorems for *ATIS* will be presented in a separate report.

The "*System Construction Theorems*" (*SCTs*), derived directly from the axioms of the *Sentential Calculus* and the intuitive creativity of a researcher or interpreter, provide a means of developing the connectedness of a system or of determining predictive outcomes. These should prove important in developing the system topology. The significance and use of *SCTs* will be clarified before presenting the logical schemas.

Significance of SCTs

System Construction Theorems (SCTs) provide the means to develop, enhance or further the explication of a theory. The significance is that they provide additional statements than what are found in the assumptions. Since they are statements of the theory, however, they are valid statements, they are not just any statements whimsically selected.

They may, however, be statements that are intuitively derived and thereby declared to be valid statements of the theory. As an initial example; however, consider the case where the derived statement is an axiom. As an axiom, it is a valid statement of the theory.

Consider Logical Schema 3: $\mathcal{P} \supset \mathcal{R} \vdash \mathcal{P} \supset (\mathcal{Q} \supset \mathcal{R})$.

Let $\mathcal{Q}$ be Axiom 105 of SIGGS: "If centrality increases, then toput decreases."

Then, regardless of what $\mathcal{P}$ and $\mathcal{R}$ represent, the following is valid:

$\mathcal{P} \supset \mathcal{R} \vdash \mathcal{P} \supset$ ("If centrality increases, then toput decreases" [Axiom 105] $\supset \mathcal{R}$),
where $\mathcal{P}$ and $\mathcal{R}$ are statements of the theory and $\mathcal{P} \supset \mathcal{R}$ is assumed to be valid.

<u>*For example*</u>:

Let $\mathcal{P}$ be the statement: "System complete connectivity increases"; and

Let $\mathcal{R}$ be the statement: "System feedin increases."

Then, $\mathcal{P} \supset \mathcal{R}$ is a statement of Axiom 100; and, therefore $\mathcal{P} \supset \mathcal{R}$ is valid.

Then, from our theorem we have:

> "System complete connectivity increases" ⊃ "System feedin increases" ⊢
>
> > "System complete connectivity increases" ⊃ ("If centrality increases, then toput decreases" ⊃ "System feedin increases").

The conclusion of this statement is equivalent to the following:

> > "System complete connectivity increases" ⊃ ("centrality decreases or toput decreases" ⊃ "System feedin increases").

It is probably clear that this is a non-obvious theorem; hence the value of the formal logic is established. But, what does it tell us?

This theorem provides a means to control a system. If the target system has complete connectivity increasing and system feedin increasing then the assumption of the theorem is satisfied. Now, assume that the target system is a terrorist system and that it is desired to decrease the complete connectivity. One way to accomplish this is to decrease toput and feedin. By decreasing toput and feedin under these conditions, system complete connectivity will decrease. Further, decreasing toput decreases feedin. Therefore, only one factor, toput, has to be controlled in order to achieve the objective of decreasing complete connectivity.

This analysis demonstrates several points. First, there are numerous non-obvious theorems that can be derived from a logical axiomatic analysis of the theory. Second, some of the outcomes, as with the above theorem, are counter-intuitive. In this case, the measure of complete connectivity is dependent on the potential complexity of the system, such complexity being degraded when toput is reduced. Third, the *SCTs* provide a fruitful means to analyze a system, but may require the intuitive skill of the analyst. On the other hand, where the logic is required for applications similar to SimEd, by defining certain "replacement" or "substitution" rules that will allow for selection of various properties or newly acquired data such logic can be programmed. These rules will probably have to be developed by an analyst who has a grasp of the pragmatic content of the theory.

Logical Schemas

SCTS: "System Construction Theorem Schema".

Logical Schema 0: $\mathcal{P} \supset \mathcal{Q}, \mathcal{Q} \supset \mathcal{R} \vdash \mathcal{P} \supset \mathcal{R}$ (Transitive Property of $\supset$)

Logical Schema 1: $\mathcal{P} \supset \mathcal{Q}, \mathcal{R} \supset \mathcal{Q} \vdash \mathcal{P} \vee \mathcal{R} \supset \mathcal{Q}$

Logical Schema 2: $\mathcal{P} \supset \mathcal{Q}, \mathcal{R} \supset \mathrm{S} \vdash \mathcal{P}\mathcal{R} \supset \mathcal{Q}\mathrm{S}$

Logical Schema 3: $\mathcal{P} \supset \mathcal{R} \vdash \mathcal{P} \supset (\mathcal{Q} \supset \mathcal{R})$ (SCTS)

Logical Schema 4: $\mathcal{P} \supset \mathcal{Q}, \mathcal{P} \supset \mathcal{R} \vdash \mathcal{P} \supset \mathcal{Q}\mathcal{R}$

Logical Schema 5: $\vdash \mathcal{Q} \supset \mathcal{P} .\!\equiv\!. \sim\mathcal{P} \supset \sim\mathcal{Q}$

Logical Schema 6: **If** $\mathcal{P} \supset \mathcal{Q}$, **then** $\mathcal{P} \vdash \mathcal{Q}$; and **If** $\mathcal{P} \vdash \mathcal{Q}$, **then** $\mathcal{P} \supset \mathcal{Q} :\equiv:$

$$\mathcal{P} \vdash \mathcal{Q} .\!\equiv\!. \vdash \mathcal{P} \supset \mathcal{Q}$$

"$\mathcal{P} \vdash \mathcal{Q} . \supset . \mathcal{P} \supset \mathcal{Q}$" is the ***Deduction Theorem***.

Logical Schema 7: $\vdash \sim(\sim\mathcal{P}\mathcal{P})$

Logical Schema 8: $\vdash \sim\sim \mathcal{P} .\!\equiv\!. \mathcal{P}$

Logical Schema 9: $\vdash \sim\mathcal{P} \vee \mathcal{P}$

Logical Schema 10: $\mathcal{P} \vdash \mathcal{Q} \supset \mathcal{P}\mathcal{Q}$ (SCTS)

Logical Schema 11: $\sim(\mathcal{Q}\mathcal{R}) \vdash \mathcal{R} \supset \sim\mathcal{Q}$

Logical Schema 12: $\mathcal{P} \supset \mathcal{Q} \vdash \mathcal{P}\mathcal{R} \supset \mathcal{Q}\mathcal{R}$ (SCTS)

Logical Schema 13: $\mathcal{R} \supset \mathrm{S} \vdash \mathcal{P}\mathcal{R} \supset \mathcal{P}\mathcal{S}$ (SCTS)

Logical Schema 14: $\mathcal{P}\mathcal{Q} \supset \mathcal{P} \vdash \mathcal{P} \supset (\mathcal{Q} \supset \mathcal{R})$ (SCTS)

Logical Schema 15: $\vdash \mathcal{P}\mathcal{Q} \supset \mathcal{R} .\!\equiv\!. \mathcal{P} \supset (\mathcal{Q} \supset \mathcal{R})$

Logical Schema 16: $\mathcal{P} \supset \sim\mathcal{Q} \vdash \mathcal{P} \supset (\mathcal{Q} \supset \mathcal{R})$ (SCTS)

Logical Schema 17: $\mathcal{P} \supset \sim\mathcal{R} \vdash \mathcal{P} \supset \sim(\mathcal{Q}\mathcal{R})$ (SCTS)

Logical Schema 18: $\mathcal{P} \supset \mathcal{Q}, \mathcal{P} \supset \sim\mathcal{R} \vdash \mathcal{P} \supset \sim(\mathcal{Q} \supset \mathcal{R})$

Logical Schema 19: $\mathcal{P}, \mathcal{P} \supset \mathcal{Q} \vdash \mathcal{Q}$ (Modus Ponens)

Logical Schema 20: $\sim\mathcal{Q}, \mathcal{P} \supset \mathcal{Q} \vdash \sim\mathcal{P}$ (Modus Talens)

The *ATIS* Predicate Calculus

While the *Sentential Calculus* has been well presented so as to demonstrate the usefulness of a formal logic, the *ATIS Predicate Calculus* will be only briefly discussed with what is required to understand its application to the analysis of the target theories. Unlike the *Sentential Calculus*, however, it is important to note that this *Predicate Calculus* is distinctly different from that required for mathematics or the mathematical sciences. Without going into any great discussion, the reason is that for *ATIS* only bound occurrences of x are considered since free occurrences do not have any apparent meaning within *ATIS*.

It was previously stated that the difference between the *Sentential* and *Predicate Calculi* was that the *Sentential Calculus* is concerned with the form of the aggregate statement with no concern of what is contained within the statement, whereas the *Predicate Calculus* is concerned with the logic of predicates; that is, statements and their constituent parts, as related to quantifiers—normally the universal and existential quantifiers. This extension will now be considered.

To make the transition from the *Predicate Calculus* required for the traditional mathematical sciences and that required for the mathematical science of *ATIS*, we will first consider the predicate notation. The predicate notation will take the form of a function; e.g., $\mathcal{P}$(x), where 'x' is an "unknown." If we can prove that $\mathcal{P}$(x) is valid for the unknown 'x', then we have $\vdash \mathcal{P}(x)$. If we have $\vdash \mathcal{P}(x)$ then we can replace 'x' with a variable and will conclude: $\vdash \forall x \mathcal{P}(x)$. For *ATIS*, it is assumed that all predicates are bound, and, therefore, all occurrences of x are variables and the validity-value of all predicate functions can be determined. Therefore, with respect to any occurrence of x, the task is to assert $\vdash \forall x \mathcal{P}(x)$ and determine if a proof exists.

Since Alonzo Church, in 1936, proved that there is no decision procedure for the *Predicate Calculus*, then the only affirmative conclusion that is possible concerning $\vdash \forall x P(x)$, with respect to the *Predicate Calculus*, is that it is valid. If no such affirmative conclusion can be found, then nothing more can be said concerning the validity of the statement. Further, the conclusion is even stronger. Church proved that there is no decision procedure regardless of what axioms are considered.

This is great news for the logician and for any researcher or analyst who is attempting to evaluate *ATIS* or an *EST* (*Education Systems Theory*). What Church has proved is that there will always be a need for the researcher and analyst, since the *Predicate Calculus*, and the *ATIS Predicate Calculus*, in particular, has no decision procedure, and, therefore, cannot be fully programmed. It is not asserted that the *ATIS Predicate Calculus* cannot be partially programmed, because it can be, but it cannot be completely programmed. The part that can be programmed, as seen below, is that part that results from the axioms that define the *ATIS Sentential Calculus*.

This point is worth elaborating. This researcher has been attempting to define the scope of this theory that is programmable, since such programs will clearly make the theory and any of its proprietary software products more appealing to users. As reflected by the extensive list of theorems that can be derived from the *ATIS Sentential Calculus*, numbering in the tens-of-thousands, and the numerous *Theorem Schemas* cited previously, it is seen that a very fruitful analysis of a system can be obtained.

Further, this researcher has proposed that utilizing data mining technologies can even extend the value of this fruitful analysis. That is, the theory software can be used as an interpreter of the data mining structured outcomes, thus enhancing the time-sensitive results required in a terrorist environment, and possibly in an educational environment. With this technology, it is no longer required that one must wait for a pattern to be determined by the data mining, but that the theory analysis will enhance the ability of the data mining technology to recognize patterns and predicted terrorist behavior or targets much earlier than utilizing the data mining technology alone.

That said, it must also be recognized that when evaluating a specific system, the logic is only semi-decidable. That is, an analyst can affirmatively determine, within the theory, that a theorem is valid, but cannot, under any circumstances, prove that it is not-valid. The reason for this is three-fold: (1) As soon as an empirical system is recognized, the problem for the analyst reverts to considerations within the *ATIS Predicate Calculus*; (2) Church has proved that such considerations are only semi-decidable; and (3) The reason that such problems considered in an empirical system are only semi-decidable is that one never knows if all possibilities have actually been considered in the proof. Systems, especially behavioral systems, are complex. This must be recognized and recognized as something positive. That is, the researcher and analyst have some very difficult tasks confronting them.

So, is the analyst without recourse? Not at all. Creative proofs from outside the theory are possible. If a reasoned argument can be found that can be construed as part of the logic of the meta-theory, then a particular theorem can be cited as being not-valid. Once the theorem has been proved in the meta-theory as being not-valid, one is then justified by claiming that the theorem is not-valid within the theory. The significance of this is that the results of this proof can then be inserted into the theory as though it had been proved within the theory.

This researcher has previously cautioned against inserting theorems directly into a computer program that has been developed as a model of the theory. However, that precaution was with respect to the *ATIS Sentential Calculus*. The *ATIS Predicate Calculus* is an entirely different matter. Whereas the theorems of the *ATIS Sentential Calculus* can be obtained directly from the axioms and, therefore, do not warrant the arbitrary insertion of theorems, the same cannot be said for the *ATIS Predicate Calculus*. Further, there will be additional work for the researcher and analyst once the initial logic has been developed and implemented for theory model applications. There are additional analyses that can be made with respect to empirical systems. It is intended that the structural properties of a system can be recognized as the topology of the system, and that the power of mathematical topology can be modified, as the *Predicate Calculus* has been, in such a way that the power of a modified mathematical topology can be used to assist in the analysis of a system. Such analyses may also have to be performed by a researcher or analyst directly with little or no reliance on a computer program. These results also will have to be manually inserted into any computer program that has been designed for a particular system.

What this simply means is that researchers and analysts of behavioral systems will always have a job. To this researcher, that is something to look forward to.

As noted previously, due to the nature of the target theories, there will be no need to distinguish between "free" and "bound" occurrences of 'x', since, without any loss of generality, all occurrences of 'x' are considered to be bound. In view of this, we have the following axioms:

(1) $\mathcal{P} \supset \mathcal{PP}$

(2) $\mathcal{PQ} \supset \mathcal{P}$

(3) $\mathcal{P} \supset \mathcal{Q} :\vee: \sim(\mathcal{QR}) \supset \sim(\mathcal{RP})$

(4) $\forall x(\mathcal{P} \supset \mathcal{Q}) :\supset: \forall x\mathcal{P} \supset \forall x\mathcal{Q}$

(5) $\mathcal{P} \supset \forall x\mathcal{P}$

(6) $\forall x\mathcal{P}(x,y) \supset \mathcal{P}(y,y)$

It should be recognized that the first three axioms are simply taken from the *Sentential Calculus*; that is, all such resulting theorems are still valid in the *Predicate Calculus*.

As seen from the axioms, the only quantifier is the universal quantifier. The existential quantifier will be defined in terms of the universal:

$$\exists x\mathcal{P} =_{df} \sim\forall x\sim\mathcal{P}$$

From this definition, we have the following equivalences:

$\vdash \exists x\mathcal{P} \equiv \sim\forall x\sim\mathcal{P}$

$\vdash \forall x\mathcal{P} \equiv \sim\exists x\sim\mathcal{P}$

$\vdash \sim\exists x\mathcal{P} \equiv \forall \sim\mathcal{P}$

$\vdash \sim\forall x\mathcal{P} \equiv \exists x\sim\mathcal{P}$

There are special conditions for ∃ for which additional notations are desired. These are the conditions in which there is exactly one x for which P is valid and when there are n x's for which P is valid. These notations are as follows:

$\exists^1 x\mathcal{P}(x)$ denotes that there is only one x for which $\mathcal{P}(x)$ is valid; and

$\exists^n\mathcal{P}(x)$ denotes that there are exactly n x's for which $\mathcal{P}(x)$ is valid.

In addition to the two quantifiers, ∀ and ∃, there are two additional quantifiers, one that will be used to specify a single component and one that will specify a class of components. These quantifiers are the descriptor quantifier, ι, and the class quantifier, ŵ. '$\iota x\mathcal{P}(x)$' is read, "the x such that P(x)"; and '$\hat{w}\mathcal{P}(w)$' is read "the class of w determined by $\mathcal{P}(w)$." These are defined as follows:

$\iota x\mathcal{P}(x) =_{df} \exists^1 x\mathcal{P}(x)$; and

$\hat{w}\mathcal{P}(w) =_{df} \iota\alpha\forall w(w \in \alpha \equiv \mathcal{P}(w))$

'$\iota x\mathcal{P}(x)$' is the *name* of the unique object that makes $\mathcal{P}(x)$ valid.

The class quantifier gives a convenient means for defining a universal class and a null class. If there are no w's in $\mathcal{P}$, then $\hat{w}\mathcal{P}$ designates the universal class, $\mathfrak{U}$, if $\mathcal{P}$ is valid, and the null class, Ø, if $\mathcal{P}$ is not-valid.

An important clarification needs to be made concerning the meaning of 'quantifier'. A logical quantifier designates a qualification of a class by indicating the logical quantity; that is, the specific components to which the qualification applies. '$\mathcal{P}$' or '$\mathcal{P}$(x)' is the *scope* of the quantification; that is, the scope of what is qualified. This will be a frequently used concept in the analysis of systems. The '*Logistic Qualifiers*' are those predicates that will be used to quantify a specific set. For example, *Toput* becomes *Input* as the result of quantifying *Toput* with respect to the *Logistic Qualifiers*. This system transition function is defined as follows:

$\sigma:(T_{\wp} \times \mathscr{L}_{i=1:n}(\mathcal{P}_i(w \in T_{\wp})) \rightarrow I_{\wp}) = \hat{w}_{I_{\wp}}\mathcal{P}(w_{T_{\wp}})$

where 'i=1:n' designates "i varies from 1 to n," and '$\mathcal{P}_i(w \in T_{\wp})$' is a qualifying statement in $\mathscr{L}$ with respect to w in $T_{\wp}$.

'$\hat{w}_{I_{\wp}}\mathcal{P}(w_{T_{\wp}})$' designates the *Input Class* determined by the *Toput Class* qualified by the $\mathcal{P}$(w)'s in $\mathscr{L}$ that make $\mathcal{P}(w_{T_{\wp}})$ valid.

An equivalent notation for $\hat{w}\mathcal{P}$ is {w | $\mathcal{P}$}, which is frequently used in mathematics.

Theory Building

In the preceding sections, the need and requirements for an axiomatic logic have been presented. In that discussion the problems relating to theory building that relies on induction, hypothetico-deductive and grounded methodologies were discussed. Now we will consider some specific concerns relating to theory building itself. What follows will be a discussion of several specific points and how to determine if theory building is actually being pursued, and if it is, what one must look for in that theory building and how to validate the theory once it is developed.

First, we will consider how to determine if the validation of a hypothesis is theoretically sound. The basic test is simply to ask the following question:

> Was the hypothesis derived from a theory that is comprehensive, consistent and complete; and, if so, is the theory axiomatic?

With respect to the requirement that the theory be axiomatic, it is simply a recognition that only axiomatic theories have been found to provide the rigorous analyses required to obtain confidence in the theory results. If an axiomatic theory cannot be obtained, then the results can always be questioned either with respect to the validation process or with respect to the "underlying assumptions" that are not stated in the theory. Descriptive and statistical-based theories can never be individually predictive and any results can always be questioned with respect to the descriptive theory, and statistical-based theories, by definition, can never be individually predictive.

Put another way, simply ask yourself:

> Was the hypothesis derived from theory? If so, what is it?

Once the theory has been established, then the next question that needs to be addressed concerns the logical basis of the theory. Most often it will be founded on a *Predicate Calculus*. If so, then there are additional questions that relate to that logic.

Any theorems that are derived from the *Predicate Calculus* are a result of the form of the theorems and not their content. The theorems of the theory that are derived directly from the basic logic are true because of their logical structure, and not at all because of their content.

In addition to theorems that are derived from the basic logic, there will be theorems that are derived from *ATIS-axioms*. Further, there will be theorems that are derived from the axioms obtained as a result of the specific empirical system being considered. Axioms and theorems from the latter two will depend upon the meaning of the terms employed within the theory or system, and not due only to their logical structure.

Class Calculus

Before considering the axioms of *ATIS* and how to develop axioms for specific systems, the axioms of the *Predicate Calculus* will be extended to include the *Class Calculus*. For this extension, a more precise and formal development of the basic logic will be presented so that a clear definition of *term*, *statement* and *formula* can be obtained.

Stratified statements determine classes. However, for *ATIS*, the initial partitioning of the system components and the definition of the system affect relations determine the stratification. Affect relations are, by definition, one class or type higher than the system components, and there is, therefore, no confusion of types.

A statement is determined by the following symbols:

$\sim \ \wedge \ \forall \ \iota \ \hat{} \ \in$ "*variables*": $x_1 \ x_2 \ \ldots \ x_n \ x \ y \ \alpha \ \beta$ "*statements*:" $\mathcal{P} \ \mathcal{Q} \ \mathcal{R}$ $\mathcal{P}(x) \ \mathcal{P}(y) \ \mathcal{P}(\iota yQ) \ \mathcal{P}(x,y) \ \mathcal{P}(y,y)$

Following are the definitions of '*term*', '*statement*' and '*formula*'. Due to their use in *ATIS*, all variables are *bound*.

(1) *term* $=_{df}$

(i) $x_1 \ x_2 \ \ldots \ x_n \ x \ y$; where "…" has its accepted meaning

(ii) $\iota x\mathcal{P}$

(iii) $\hat{w}\mathcal{P}$

(2) *statement* $=_{df}$

(i) $A \in B$, where 'A' and 'B' are terms, 'A' is a *component* and 'B' is a *class*, since only sentences concerned exclusively with classes are considered to be statements

(ii) $\forall x\mathcal{P}$, where 'x' is a variable and '$\mathcal{P}$' is a statement

(iii) $\sim\mathcal{P}$, where '$\mathcal{P}$' is a statement

(iv) $\mathcal{P} \wedge \mathcal{Q}$, where '$\mathcal{P}$' and '$\mathcal{Q}$' are statements

(3) *formula* $=_{df}$

(i) $\mathcal{S}$, where '$\mathcal{S}$' is comprised of a sequence of statements constructed with '$\sim$' and '$\wedge$'

(4) $\alpha = \beta \ =_{df} \forall x(x \in \alpha \equiv x \in \beta) =_{df} \alpha = \beta =_{df} \alpha =_x \beta$

'$\mathcal{P}$' is referred to as the scope of the quantifiers.

Definition (4) defines equality of sets. The last notation, $\alpha =_x \beta$, is very useful in *ATIS*. Due to the complexity of systems, it may be that various properties are defined with respect to the same set of components. Rather than having to consider numerous sets, the properties can be defined with respect to a specific subset. For example, one may wish to determine the behavior of a system with respect to various subsets. Such can be designated as follows: $\mathbb{S} =_B \mathcal{V}$; $\mathbb{S} =_C \mathcal{L}$; and $\mathbb{S} =_D \mathcal{G}$. Then an APT analysis can be performed on the following set: $\ell = \{\mathcal{V}, \mathcal{L}, \mathcal{G}\}$.

With the foregoing definitions, we now have the following axiom schemas extended from the *Predicate Calculus* to include the axiom schema for the *Class Calculus*, Axiom (12).

Transformation Rule, Modus Ponens: $\mathcal{P}, \mathcal{P} \supset \mathcal{Q} \vdash \mathcal{Q}$

(1) $\mathcal{P} \supset \mathcal{P}\mathcal{P}$

(2) $\mathcal{P}\mathcal{Q} \supset \mathcal{P}$

(3) $(\mathcal{P} \supset \mathcal{Q}) \vee [\sim(\mathcal{Q}\mathcal{R}) \supset \sim(\mathcal{R}\mathcal{P})]$

(4) $\forall x(\mathcal{P} \supset \mathcal{Q}) \supset (\forall x\mathcal{P} \supset \forall x\mathcal{Q})$

(5) $\mathcal{P} \supset \forall x\mathcal{P}$

(6) $\forall x\mathcal{P}(x,y) \supset \mathcal{P}(y,y)$

   i. $\forall x\mathcal{P}(x) \supset \mathcal{P}(y)$

(7) $\forall x,y,z[(x = y) \supset (x \in z \supset y \in z)]$

(8) $\forall x_1,x_2,\ldots,x_n(\forall x\mathcal{P}(x) \supset \mathcal{P}(\iota y\mathcal{Q})$

(9) $\forall x_1,x_2,\ldots,x_n[\forall x(\mathcal{P} \equiv \mathcal{Q}) \supset (\iota x\mathcal{P} = \iota x\mathcal{Q})]$

(10) $\forall x_1,x_2,\ldots,x_n[\iota x\mathcal{P}(x) \equiv \iota y\mathcal{P}(y)]$

   i. $\forall x_1,x_2,\ldots,x_n[\iota x\mathcal{P} = \iota y\mathcal{Q}]$

(11) $\forall x_1,x_2,\ldots,x_n[\exists^1 x\mathcal{P} \supset (\forall x[\iota x\mathcal{P} = x \equiv \mathcal{P}])]$

   i. $\forall x_1,x_2,\ldots,x_n[\exists^1 x\mathcal{P}(x) \supset (\forall x[\iota x\mathcal{P}(x) = x \equiv \mathcal{P}(x)])]$

   ii. $\forall x_1,x_2,\ldots,x_n[\exists^1 x\mathcal{P}(x) \supset (\forall y[\iota x\mathcal{P}(x) = y \equiv \mathcal{P}(y)])]$

(12) $\exists y\forall x(x \in y \equiv \mathcal{P})$

Axiom Schemas (1) to (3) are the valid-value axioms of the *Sentential Calculus*.

Axiom Schemas (4) to (6) allow for generalization from $\vdash\mathcal{P}$ to $\vdash\forall x\mathcal{P}$.

Axiom Schema (6) provides for substitution of a value for a variable.

Axiom Schema (7) allows for substitution of equivalent terms resulting from equality.

Axiom Schemas (8) to (11) are the axiom schemas for ι.

Axiom Schema (8) asserts that if $\mathcal{P}$(x) is true for all x, then ιy$\mathcal{Q}$ is the name of one of those objects.

Axiom Schema (9) asserts that if $\mathcal{P}$ and $\mathcal{Q}$ are equivalent for all x, then ιx$\mathcal{P}$ and ιx$\mathcal{Q}$ are names of the same object.

Axiom Schema (10) allows for change of variables.

Axiom Schema (11) asserts that if there is a unique x that makes $\mathcal{P}$ valid, then ιx$\mathcal{P}$ is that x.

Axiom Schema (12) is the schema that introduces classes. This axiom allows for the *Set Calculus* to be integrated into the formal theory.

Relation Calculus

For *ATIS*, we are concerned with attempting to use as many mathematical constructs as possible while clearly describing the desired system properties.

While mathematics is frequently concerned with *functions*, for *ATIS* the concerns may be directed more toward *relations*.

However, while functions are normally considered as being single-valued, many-valued functions are possible. The relation $[\![x,x^2]\!]$ is a multi-valued function. $[\![x,x<y]\!]$ also is a multi-valued function. These are well-defined functions since the ordered pairs that define the functions are well-defined. Whether or not these are considered functions or relations is not clear; that is, there does not seem to be any clear distinction between the two. With '*function*' being restricted to *single-valued functions*, these examples would be considered as *relations*. One distinction has been that '*function*' was restricted to relations that resulted in well-defined curves, whereas '*relation*' would be for those statements that defined all other characterizations. Thus, $[\![x,x^2]\!]$ would be a *function*, and $[\![x,x<y]\!]$ would be a *relation*.

In *ATIS*, the distinction between '*function*' and '*relation*' will not be considered. The only question is whether or not the appropriate mathematical construct clearly portrays the system characteristic being considered. It appears as though most of the concerns for *ATIS* will be with respect to *morphisms*; that is, *relational mappings*. Whether such mappings are '*functions*' or '*relations*' is moot. If a single-valued function is required, then such can be stated. For purposes of analysis, morphisms or relations will be considered, since functions are a special type of relation. Further, where the "function notation" is used, it is not to be construed as restrictive. Normally, it will probably designate a single-valued function, but such in this theory is not required. Either the context or by definition, the type of function will be determined.

The *Relation Calculus* for *ATIS* is concerned with the affect relations that define a system and the morphisms that characterize the properties of the system as derived from those affect relations.

The *Relation Calculus* axiom schemas will be presented first. This will complete the presentation of the formal logic.

Following the presentation of the formal logic, the content required for a *General System Theory* will be introduced. First, the axiom that asserts the existence of a *General System* will be introduced. Then the axioms that establish the empirical systems that are to be analyzed and the criteria for such analysis will be given.

We have already introduced the notation that will be used to identify a class or set of objects, or components, $\hat{w}\mathcal{P}(w)$. Now the characterization of those components will be discussed.

Whereas x identifies a single component within the set, it may be that we wish to identify an object that consists of two or more components. The following notations will be used to identify such sets.

'{x, y}' identifies a component of a set that consists of two single components.

If it is desired to specify that the set consists only of binary-components, then the following notation will so indicate: $\hat{w}^2\mathcal{P}(w)$. This notation designates that the class or set of components consists only of sets each of which contains two single components.

Hence, '$\hat{w}^2\mathcal{P}(w)$' designates a *family* of binary sets.

By extension, '$\hat{w}^n\mathcal{P}(w)$' designates a family of sets, each member of which contains n components; that is, $\{x_1, x_2, \ldots, x_n\}_i \in w$.

For affect relations, an additional type of set will be required. This set will contain binary-components and a set that contains one of the binary-components in a unary-component set. That is, the set will be configured by the following representation:

$$\{\{x\}, \{x, y\}\}$$

Where each unary-, binary-component set is included and no other sets are included. This notation is frequently represented by the ordered pair: (x,y).

For this set, the class quantifier will be represented as: $\hat{w}^{2|1}\mathcal{P}(w)$.

By extension, '$\hat{w}^{n|n-1|\ldots|1}\mathcal{P}(w)$' designates a family of sets that include all and only those ordered subsets of the largest set. For n = 4, the family of sets would be characterized by components of the form: {{a}, {a,b}, {a,b,c}, {a,b,c,d}} = (a,b,c,d), an **ordered** 4-tuple.

With the foregoing definitions, we now have the following axiom schema for the *Relation Calculus*, Axiom (13), which introduces relations.

Axiom (13) $\exists z \forall x,y(\{x,y\} \in z \wedge \{x\} \in z \equiv \mathcal{R}$

The following axiom schemas provide for substitution within and identification of relations.

Axiom (14) $\forall x\mathcal{R}(x) \supset \mathcal{R}(\iota y\mathcal{Q})$

Axiom (15) $\forall x(\mathcal{R} \equiv \mathcal{Q}) \supset (\iota x\mathcal{R} \equiv \iota x\mathcal{Q})$

Axiom (16) $\iota(x,y)\mathcal{R}(x,y) \equiv \iota(p,q)\mathcal{R}(p,q))$

Axiom Schema (14) asserts that if $\mathcal{R}(x)$ is true for all x, then $\iota y\mathcal{Q}$ is the name of one of those relations.

Axiom Schema (15) asserts that if relations $\mathcal{R}$ and $\mathcal{Q}$ are equivalent for all x, then $\iota x\mathcal{R}$ and $ix\mathcal{Q}$ are names of the same relation. This is a critical axiom for determining morphisms.

Axiom Schema (16) allows for change of variables.

ATIS Calculus

In the preceding sections, the formal logic has been established.

Now, the calculus must be developed that begins to provide the substance for the desired theories. These theories are descriptive of what will be called *General Systems*. Therefore, the first axiom will introduce the characteristics of a *General System*.

The development of the calculus that results in the empirical theory is dependent upon the concept of an *Options Set*. The *Options Set* is that listing of *Properties* and *Associated Axioms* that will result in a system-descriptive theory that will be analyzed pursuant to the derived formal logic herein established. The specific *Options Set* herein developed is the ***ATIS* Options Set**. This set will consist of the derived list of properties, and all *General System* axioms that are associated with those properties.

An analysis of a system is obtained by determining those properties that are descriptive of the system. Those properties are then identified in the ***ATIS* Options Set**. Following identification of these properties, the *Associated Axioms* are then selected. *Associated Axioms* are those in which one or more of the selected properties occur. With the selection of these axioms, an analysis of the system is possible using the *Predicate* and *Relation Calculi* herein developed.

It is also intended that a topological analysis will eventually be possible either by the direct use of operations taken from mathematical topology or a derivation thereof. Such an analysis, along with other analytic techniques, is beyond the scope of this report.

The following axiom asserts that if we have a set of a specific system defined by components and a set of that system defined by relations of those components, then we have a *General System* that can be characterized by *ATIS*. Axiom Schema (17) is the *General System* axiom scheme.

Axiom (17) $\hat{w}\mathcal{P}(w) \equiv \mathbb{S}_x \wedge \hat{y}\mathcal{P}(y) \equiv \mathbb{S}_\phi \equiv \mathcal{G}(\mathbb{S}_x,\mathbb{S}_\phi)$

The affect relations, $\mathbb{S}_\phi$, determine the properties, $\mathcal{P}$, of an *Intentional General System*.

Axiom (18) asserts that if we have an *Intentional General System*, $\mathcal{IG}$, then for every property, $\mathcal{P}$, there exists a property qualifier that determines the class $\hat{w}\mathcal{P}(w)$.

Axiom (18) $\mathcal{IG} \supset \forall\mathcal{P}(w)\exists\hat{w}(\hat{w}\mathcal{P}(w))$

The following axiom asserts that if we have a property class then there is a morphism that can be defined between that class and another property class.

Axiom (19) $\hat{w}\mathcal{P}(w) \supset \exists\mathcal{X}\exists\hat{y}\mathcal{P}(y)(\mathcal{X}(\hat{w}\mathcal{P}(w) \rightarrow \hat{y}\mathcal{P}(y)))$

ATIS Options Set Defined

The *ATIS Options Set*, $\mathcal{A}$, is defined by the set of system properties, $\mathcal{P}$, and system Affect Relations, $\mathcal{A}$.

$\mathcal{A} =_{df} \hat{w}_{i=1:n}\mathcal{P}_i(w_i) \cup \hat{y}_{j=1:n}\mathcal{A}_i(y_j)$

Definitions of Logical Operations in Proofs

In addition to the logical schemas presented above, some proofs of theorems may require an application of the definition of the logical operations. An example is given below.

The following operations were previously defined:

Definition. $\mathcal{P} \vee \mathcal{Q} =_{df} \sim(\sim\mathcal{P} \wedge \sim\mathcal{Q})$

Definition. $\mathcal{P} \supset \mathcal{Q} =_{df} \sim(\mathcal{P} \wedge \sim\mathcal{Q})$

Definition. $\mathcal{P} \equiv \mathcal{Q} =_{df} (\mathcal{P} \supset \mathcal{Q}) \wedge (\mathcal{Q} \supset \mathcal{P})$.≡. $\sim(\mathcal{P} \wedge \sim\mathcal{Q}) \wedge \sim(\mathcal{Q} \wedge \sim\mathcal{P})$

The following theorem demonstrates an application of the use of definitions in the proof of a theorem.

Theorem. $\vdash {}_{HO}\mathscr{C}^{c\uparrow} \vee {}_{F}\mathscr{C}^{c\downarrow} \supset {}_{S}\mathscr{C}^{c\downarrow}$ "If hierarchical order is constant or increasing, or if flexibility is constant or decreasing, then strongness is constant or decreasing."

Proof:

1. ${}_{S}\mathscr{C}^{\uparrow} \supset {}_{HO}\mathscr{C}^{\downarrow}$ Axiom 55; i.e., "If strongness increases, then hierarchical order decreases."
2. ${}_{S}\mathscr{C}^{\uparrow} \supset {}_{F}\mathscr{C}^{\uparrow}$ Axiom 56; i.e., "If strongness increases, then flexibility increases."
3. ${}_{S}\mathscr{C}^{\uparrow} \supset {}_{HO}\mathscr{C}^{\downarrow} \wedge {}_{F}\mathscr{C}^{\uparrow}$ Logical Schema 4
4. $\sim({}_{HO}\mathscr{C}^{\downarrow} \wedge {}_{F}\mathscr{C}^{\uparrow}) \supset \sim {}_{S}\mathscr{C}^{\uparrow}$ Logical Schema 5
5. ${}_{HO}\mathscr{C}^{c\uparrow} \vee {}_{F}\mathscr{C}^{c\downarrow} \supset \sim {}_{S}\mathscr{C}^{\uparrow}$ Definition of '$\vee$'
6. ${}_{HO}\mathscr{C}^{c\uparrow} \vee {}_{F}\mathscr{C}^{c\downarrow} \supset {}_{S}\mathscr{C}^{c\downarrow}$ Logical Equivalence of '~'
7. $\vdash {}_{HO}\mathscr{C}^{c\uparrow} \vee {}_{F}\mathscr{C}^{c\downarrow} \supset {}_{S}\mathscr{C}^{c\downarrow}$ Q.E.D.[6]

[6] "Q.E.D." comes from the Latin *quod erat demonstrandum*, "that which was to be demonstrated"; or, in mathematics, "that which was to be proved."

Axiom 181 is a Theorem

With the number of axioms presented for the theory, it is possible that some of the axioms are in fact theorems; that is, they are derivable from the other axioms. Such is the case with Axiom 181.

Axiom 181 states: $\mathcal{Z}^{\uparrow} \wedge \mathcal{X}^{+c} \supset {}_{C}\mathscr{C}^{\downarrow}$.

That is: "If size increases and complexity growth is constant, then centrality decreases."

This statement will now be proved as a theorem.

Theorem 181. $\vdash \mathcal{Z}^{\uparrow} \wedge \mathcal{X}^{+c} \supset {}_{C}\mathscr{C}^{\downarrow}$

Proof:

1. $\mathcal{Z}^{\uparrow} \wedge \mathcal{X}^{+c} \supset \mathrm{T}_{\mathfrak{p}}^{\uparrow}$ — Axiom 194; i.e., "If size increases and complexity growth is constant, then toput increases."
2. $\mathrm{T}_{\mathfrak{p}}^{\uparrow} \supset {}_{C}\mathscr{C}^{\downarrow}$ — Axiom 90; i.e., "If toput increases, then centrality decreases."
3. $\mathcal{Z}^{\uparrow} \wedge \mathcal{X}^{+c} \supset {}_{C}\mathscr{C}^{\downarrow}$ — Logical Schema 0 (Transitive Property) on Steps 1 and 2
4. $\vdash \mathcal{Z}^{\uparrow} \wedge \mathcal{X}^{+c} \supset {}_{C}\mathscr{C}^{\downarrow}$ — Q.E.D.

Inconsistent Axioms

Certain axioms of the SIGGS Theory have or will be found to be inconsistent. That is, they are inconsistent when combined within the same theory. This does not mean that either axiom is “wrong” or “false” or “not-valid.” It simply means that they cannot be taken together in the same theory. No determination will be made at this time as to which axiom is more appropriate for the theory. It may simply be that several theories will be developed from the SIGGS Theory axioms.

The following pairs of axioms have been found to be inconsistent and will be so proved below: Axioms 55 and 112, Axioms 90 and 106, and Axioms 175 and 183.

Theorem 55-112. $\vdash \sim({}_S\mathscr{C}^{\uparrow} \supset {}_{HO}\mathscr{C}^{\downarrow} :\equiv: {}_S\mathscr{C}^{\uparrow} \wedge {}_{HO}\mathscr{C}^{c} \supset {}_R\mathscr{R}^{\downarrow})$

That is: “$\vdash$ ~(Axiom 55 :≡: Axiom 112)”

Proof:

1. ${}_S\mathscr{C}^{\uparrow} \supset {}_{HO}\mathscr{C}^{\downarrow}$ — Axiom 55; i.e., “If strongness increases, then hierarchical order decreases.”
2. ${}_S\mathscr{C}^{\uparrow} \vdash {}_{HO}\mathscr{C}^{\downarrow}$ — Deduction Theorem on 1
3. ${}_S\mathscr{C}^{\uparrow}$ — Assumption from 2
4. ${}_{HO}\mathscr{C}^{\downarrow}$ — Modus Ponens on 3 and 1
5. ${}_S\mathscr{C}^{\uparrow} \wedge {}_{HO}\mathscr{C}^{c} \supset {}_R\mathscr{R}^{\downarrow}$ — Axiom 112; i.e., “If strongness increases and hierarchical order is constant, then regulation decreases.”
6. ${}_S\mathscr{C}^{\uparrow} \supset. {}_{HO}\mathscr{C}^{c} \supset {}_R\mathscr{R}^{\downarrow}$ — Logical Equivalence 3 on 5
7. ${}_S\mathscr{C}^{\uparrow} \vdash {}_{HO}\mathscr{C}^{c} \supset {}_R\mathscr{R}^{\downarrow}$ — Deduction Theorem on 6
8. ${}_S\mathscr{C}^{\uparrow}, {}_{HO}\mathscr{C}^{c} \vdash {}_R\mathscr{R}^{\downarrow}$ — Deduction Theorem on 7
9. ${}_{HO}\mathscr{C}^{c}$ — Assumption from 8
10. ${}_{HO}\mathscr{C}^{\downarrow} \wedge {}_{HO}\mathscr{C}^{c}$ — From Steps 4 and 9
11. ${}_{HO}\mathscr{C}^{\downarrow} \wedge {}_{HO}\mathscr{C}^{c}$ — Contradiction of ‘$\wedge$’
12. $\vdash \sim({}_S\mathscr{C}^{\uparrow} \supset {}_{HO}\mathscr{C}^{\downarrow} :\equiv: {}_S\mathscr{C}^{\uparrow} \wedge {}_{HO}\mathscr{C}^{c} \supset {}_R\mathscr{R}^{\downarrow})$ — Q.E.D.

Theorem 90-106. $\vdash \sim(T_{\wp}^{\uparrow} \supset {}_{C}\mathscr{C}^{\downarrow} :\equiv: {}_{C}\mathscr{C}^{\uparrow} \vee {}_{S}\mathscr{C}^{\uparrow} \supset T_{\wp}^{\uparrow})$

That is: "⊢ ~(Axiom 90 :≡: Axiom 106)"

Proof:

1. $T_{\wp}^{\uparrow} \supset {}_{C}\mathscr{C}^{\downarrow}$ — Axiom 90; i.e., "If toput increases, then centrality decreases."
2. $T_{\wp}^{\uparrow} \vdash {}_{C}\mathscr{C}^{\downarrow}$ — Deduction Theorem on 1
3. $T_{\wp}^{\uparrow}$ — Assumption from 2
4. ${}_{C}\mathscr{C}^{\downarrow}$ — Modus Ponens on 3 and 1
5. ${}_{C}\mathscr{C}^{\uparrow} \vee {}_{S}\mathscr{C}^{\uparrow} \supset T_{\wp}^{\uparrow}$ — Axiom 106; i.e., "If complete connectivity increases or strongness increases, then toput increases."
6. ${}_{C}\mathscr{C}^{\uparrow} \supset T_{\wp}^{\uparrow}$ — Assumption of Case on 5
7. ${}_{C}\mathscr{C}^{\uparrow} \vdash T_{\wp}^{\uparrow}$ — Deduction Theorem on 6
8. ${}_{C}\mathscr{C}^{\uparrow}$ — Assumption from 7
9. ${}_{C}\mathscr{C}^{\downarrow} \wedge {}_{C}\mathscr{C}^{\uparrow}$ — From Steps 4 and 8
10. ${}_{C}\mathscr{C}^{\downarrow} \wedge {}_{C}\mathscr{C}^{\uparrow}$ — Contradiction of '∧'
11. $\vdash \sim(T_{\wp}^{\uparrow} \supset {}_{C}\mathscr{C}^{\downarrow} :\equiv: {}_{C}\mathscr{C}^{\uparrow} \vee {}_{S}\mathscr{C}^{\uparrow} \supset T_{\wp}^{\uparrow})$ — Q.E.D.

Theorem 175-183. $\vdash \sim(\mathcal{X}^{-\uparrow} \supset \mathcal{Z}^{-\uparrow} \vee {}_{D}\mathscr{C}^{\uparrow} :\equiv: \mathcal{Z}^{\downarrow} \wedge \mathcal{X}^{-\uparrow} \supset {}_{D}\mathscr{C}^{\downarrow})$

That is: "$\vdash$ ~(Axiom 175 :≡: Axiom 183)"

Proof:

1.	$\mathcal{X}^{-\uparrow} \supset \mathcal{Z}^{-\uparrow} \vee {}_{D}\mathscr{C}^{\uparrow}$	Axiom 175; i.e., "If complexity degeneration increases, then size degeneration increases or disconnectivity increases."
2.	$\mathcal{X}^{-\uparrow} \vdash \mathcal{Z}^{-\uparrow} \vee {}_{D}\mathscr{C}^{\uparrow}$	Deduction Theorem on 1
3.	$\mathcal{X}^{-\uparrow}$	Assumption from 2
4.	$\mathcal{Z}^{-\uparrow} \vee {}_{D}\mathscr{C}^{\uparrow}$	Modus Ponens on 3 and 1
5.	${}_{D}\mathscr{C}^{\uparrow}$	Assumption of Case on 4
6.	$\mathcal{Z}^{\downarrow} \wedge \mathcal{X}^{-\uparrow} \supset {}_{D}\mathscr{C}^{\downarrow}$	Axiom 183; i.e., "If size decreases and complexity degeneration increases, then disconnectivity decreases."
7.	$\mathcal{Z}^{\downarrow} \wedge \mathcal{X}^{-\uparrow} \vdash {}_{D}\mathscr{C}^{\downarrow}$	Deduction Theorem on 4
8.	$\mathcal{Z}^{\downarrow} \wedge \mathcal{X}^{-\uparrow}$	Assumption from 5
9.	${}_{D}\mathscr{C}^{\downarrow}$	Modus Ponens on 6 and 4
10.	${}_{D}\mathscr{C}^{\uparrow} \wedge {}_{D}\mathscr{C}^{\downarrow}$	Steps 5 and 9
11.	${}_{D}\mathscr{C}^{\uparrow} \wedge {}_{D}\mathscr{C}^{\downarrow}$	Contradiction of '$\wedge$'
12.	$\vdash \sim(\mathcal{X}^{-\uparrow} \supset \mathcal{Z}^{-\uparrow} \vee {}_{D}\mathscr{C}^{\uparrow} :\equiv: \mathcal{Z}^{\downarrow} \wedge \mathcal{X}^{-\uparrow} \supset {}_{D}\mathscr{C}^{\downarrow})$	Q.E.D.